$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\underbrace{(\vec{a}\cdot\vec{c})}_{\text{Coefficient of } \vec{b}} - \vec{c}\underbrace{(\vec{a}\cdot\vec{b})}_{\text{Coefficient of } \vec{c}}$$

$\vec{a}\cdot\vec{c} = 0 \Rightarrow \vec{a} \perp^{r} \vec{c}$ (90°)

$$-(\vec{a}\cdot\vec{b}) = \frac{\sqrt{3}}{2}$$

$$ab\cos\theta = -\frac{\sqrt{3}}{2}$$

$$\cos\theta = -\frac{\sqrt{3}}{2}$$

$$\cos\theta = \cos 150$$

$$\boxed{\theta = 150^\circ}$$

So $150^\circ, 90^\circ$ so option (b) is correct.

* SCALAR PRODUCT OF FOUR VECTORS:-

Prove the identity -

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

Proof:- $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b})\cdot\vec{r} = \vec{a}\cdot(\vec{b}\times\vec{r})$

dot and cross can be interchanged.

Put $\vec{c}\times\vec{d} = \vec{r}$

$$= \vec{a}\cdot[\vec{b}\times(\vec{c}\times\vec{d})] = \vec{a}\cdot[(\vec{b}\cdot\vec{d})\vec{c} - (\vec{b}\cdot\vec{c})\vec{d}]$$

$$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

Proved

Q. If $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} - 4\hat{k}$, $\vec{c} = \hat{i} + \hat{j} + \hat{k}$ find $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c})$

Sol^n

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ -1 & 2 & -4 \end{vmatrix} = \hat{i}(-12+2) - \hat{j}(-8+1) + \hat{k}(4+3)$$

$$\vec{a}\times\vec{b} = -10\hat{i} + 9\hat{j} + 7\hat{k}$$

$$\vec{a}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \hat{i}(3+1) - \hat{j}(2+1) + \hat{k}(2-3)$$

$$\vec{a}\times\vec{c} = 3\hat{i} - 3\hat{j} - \hat{k}$$

So $(\vec{a}\times b)\cdot(\vec{a}\times\vec{c}) = (-10\hat{i} + 9\hat{j} + 7\hat{k})\cdot(4\hat{i} - 3\hat{j} - \hat{k})$

$$= -40 - 27 - 7$$
$$= -74 \quad \text{Ans}$$

* VECTOR PRODUCT OF FOUR VECTORS :-

$$\boxed{(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})}$$

Put $\vec{a}\times\vec{b} = \vec{r}$

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{r}\times(\vec{c}\times\vec{d})$$
$$= (\vec{r}\cdot\vec{d})\vec{c} - (\vec{r}\cdot\vec{c})\vec{d}$$
$$= [(\vec{a}\times\vec{b})\cdot\vec{d}]\vec{c} - [(\vec{a}\times\vec{b})\cdot\vec{c}]\vec{d}$$
$$= [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d} \quad \text{--- (1)}$$

$\therefore (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ lies in the plane of $\vec{c}$ and $\vec{d}$.

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b})\times\vec{s} \qquad \text{put } \vec{c}\times\vec{d} = \vec{s}$$
$$= -\vec{s}\times(\vec{a}\times\vec{b})$$
$$= -(\vec{s}\cdot\vec{b})\vec{a} + (\vec{s}\cdot\vec{a})\vec{b}$$

$$= -[(\vec{c}\times\vec{d})\cdot\vec{b}]\,\vec{a} + [(\vec{c}\times\vec{d})\cdot\vec{a}]\,\vec{b}$$

$$= -[\vec{b}\,\vec{c}\,\vec{d}]\,\vec{a} + [\vec{a}\,\vec{c}\,\vec{d}]\,\vec{b}$$

$\therefore$ $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ lies in the plane of $\vec{a}$ and $\vec{b}$. — (II)

Geometrical Interpretation :-

From (1) and (2) we conclude that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ is a vector parallel to the line of intersection of the plane containing $\vec{a}, \vec{b}$ and plane containing $\vec{c}, \vec{d}$.

Q. Show that -

$$(\vec{B}\times\vec{C})\times(\vec{A}\times\vec{D}) + (\vec{C}\times\vec{A})\times(\vec{B}\times\vec{D}) + (\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = -2(\vec{A}\,\vec{B}\,\vec{C})\,\vec{D}$$

Sol^n Proof:- $(\vec{B}\times\vec{C})\times(\vec{A}\times\vec{D}) + (\vec{C}\times\vec{A})\times(\vec{B}\times\vec{D}) + (\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D})$

$$= [(\vec{B}\vec{C}\vec{D})\vec{A} - (\vec{B}\vec{C}\vec{A})\vec{D}] + [(\vec{C}\vec{A}\vec{D})\vec{B} - (\vec{C}\vec{A}\vec{B})\vec{D}] + [(-\vec{B}\vec{C}\vec{D})\vec{A} + (\vec{A}\vec{C}\vec{D})\vec{B}]$$

$$= (\vec{B}\vec{C}\vec{D})\vec{A} - (\vec{B}\vec{C}\vec{D})\vec{A} + (\vec{C}\vec{A}\vec{D})\vec{B} + (\vec{A}\vec{C}\vec{D})\vec{B} - (\vec{B}\vec{C}\vec{A})\vec{D} - (\vec{C}\vec{A}\vec{B})\vec{D}$$

$$= -(\vec{A}\vec{C}\vec{D})\vec{B} + (\vec{A}\vec{C}\vec{D})\vec{B} - (\vec{A}\vec{B}\vec{C})\vec{D} - (\vec{A}\vec{B}\vec{C})\vec{D}$$

$$= -2(\vec{A}\vec{B}\vec{C})\vec{D}$$

Proved